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A Survey of Techniques for the Analysis of Sampled - Data Systems with a Variable Sampling Rate

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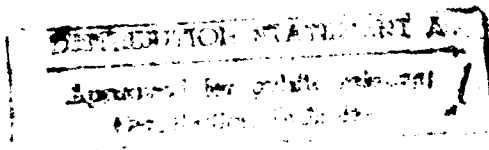
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Aeronautical Systems Division
Air Force Systems Command
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FOREWORD

The research described in this report, *A Survey of Techniques for the Analysis of Sampled-Data Systems with a Variable Sampling Rate*, by George A. Bekey, was carried out under the technical direction of C. T. Leondes and G. Estrin and is part of the continuing program in Control Systems Theory.

This project is conducted under the sponsorship of Aeronautical Systems Division, Flight Control Laboratory, Project Engineer Charles Harmon, Wright-Patterson Air Force Base. Submitted in partial fulfillment of Contract Number AF 33(616)-7139, Project Number 8225.

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ABSTRACT

This report presents several techniques which can be used for the analysis of sampled-data systems with a non-constant sampling period. It is shown that the application of z-transform techniques is limited to cases where the sampling pattern repeats periodically. Several special cases, including the cyclically-varying sampling period and the "Skipped sample" problem are outlined. The most general methods available are based on a direct solution of the system difference equations. These equations, while time consuming to solve, do make possible the evaluation of transient response sample-by-sample. The advantages, limitations and possible extensions of the various methods are outlined. The report includes a number of simple examples and an extensive bibliography.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER:



C. R. BRYAN
Technical Director
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1. INTRODUCTION

The analysis of sampled-data systems with a fixed sampling period has been treated in a large number of publications,^{16, 19} primarily by use of the z-transform method. Only a relatively small amount of attention has been devoted to the larger class of sampling systems in which the sampling interval is variable, depending either on time or on a system variable; the former being linear and the latter non-linear. Sampled-data systems with a variable sampling rate show promise of having two extremely interesting properties:

- (1) The operation of a system at its minimum sampling rate, consistent with required performance and stability criteria, leads to a class of adaptive systems in which the energy or control effort is minimized. Such control systems may be extremely important for applications when available energy is severely limited, such as satellite attitude control.
- (2) The operation of a system which adjusts its sampling rate according to performance requirements may closely resemble the behavior of a human operator and therefore may lead to clues about certain functions of the central nervous system.

The purpose of this paper is to review and summarize the major techniques described in the literature for the analysis of sampled-data systems with a non-constant sampling interval. A general and unified approach to this problem seems to be lacking. Most of the available papers concerning variable sampling assume either periodic variations in sampling rate or random variations in sampling intervals. This report presents an outline of the major methods presently available and discusses briefly their applicability and limitations.

A recent article by Jury presents a brief discussion of various types of sampling schemes.²²

1.1 Classification

Sampled data systems with a non-constant sampling rate may be divided

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into three general types:

- (1) Time-dependent sampling: The sampling intervals are given by a known function of time. Nearly all the literature on variable-sampling rate systems is concerned with this type of sampling. Specific systems include: (a) Cyclic-rate systems (where the sampling interval is a periodic function of time, i. e., a pattern of sampling intervals repeats periodically); (b) Skip-sampled systems (in which samples in an otherwise constant rate system are occasionally missed); and (c) Piecewise-constant sampling rate systems (where the sampling rate changes discontinuously after a number of samples, remaining constant between changes).
- (2) Nonlinear sampling: The sampling intervals are functions of dependent variables or system performance. Clearly, this problem is considerably more difficult than that of time dependent sampling, and very little information is available in the literature on methods of handling such problems.
- (3) Random sampling: In this type of system the sampling interval is a random variable. The synthesis of optimum filters for such systems with stationary random sampling has been treated in the literature.^{1, 11, 15}

In the present report we shall concern ourselves exclusively with deterministic sampling and omit any further discussion of random sampling. Furthermore, the report is concerned with constant pulse width systems, preferably those which can be analyzed using the impulse approximation. A number of extremely interesting papers have dealt with pulse-width modulation in sampled-data systems but this problem will be considered as being outside the scope of this discussion.

1.2 Methods of Analysis

A number of methods have been proposed for the analysis of sampled-data systems with variable sampling rate. In many cases, the methods are

applicable only to the case of periodic, time dependent sampling rates. The major techniques known to the author fall into the following classification:

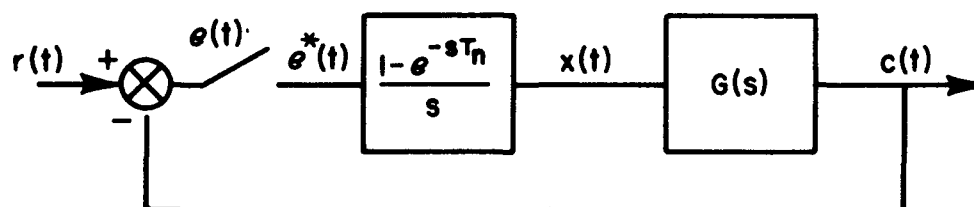
- (1) The solution of difference equations with variable coefficients in the time domain.
- (2) The use of z-transform and modified z-transforms.
- (3) The use of matrix methods for formulation of the solution.
- (4) The application of state-variable concepts for a general formulation of the problem.
- (5) Approximate solutions, using continuous system techniques to approximate sampled data systems.

The five methods above will be examined in small detail in the following paragraphs. While it is clear from the discussion which follows that there is a great deal of overlap between these five methods, they are presented separately for convenience and because they appear separately in the literature.

2. ANALYSIS USING DIFFERENCE EQUATIONS WITH TIME-VARYING COEFFICIENTS

The time-domain behavior of a sampled-data system can be described using difference equations, even if the sampling intervals are not of constant length. Thus a q-th order system can be described by q 1-st order difference equations or, by substitution, a single q-th order equation which expresses the output at any sampling instant in terms of the input and output values at the (q-1) past sampling instants.

Consider the error-sampled system of Figure 1, which includes a zero-order hold and a linear "plant" described by its transfer function $G(s)$.



ERROR-SAMPLED SYSTEM

FIGURE 1

The sampler is assumed to close at instants t_1, t_2, t_3, \dots , where the sampling intervals

$$T_n = t_{n+1} - t_n$$

are not necessarily equal. For the moment, the sampling instants will be assumed known, and the removal of this restriction will be discussed later.

The output of the system during the interval

$$t_n \leq t \leq t_{n+1}$$

is then given by the relation⁸

$$c(t) = X_n g(t-t_n) + \sum_{p=0}^{q-1} f_p(t-t_n) \left[\frac{d^{(p)} c(t)}{dt^{(p)}} \right]_{t=t_n} \quad (1a)$$

where t_n = time at the start of the (n+1) sampling interval of duration T_n

$$x_n = \text{output of zero-order hold} = e(t_n)$$

$c(t)$ = continuous system output

$g(t)$ = step function response of system $G(s)$

q = order of the system

and the $f_p(t)$ are time functions which result from the initial conditions at the start of the n-th sampling interval.

Since the system is of order q , we need q first order equations to completely describe it. The additional equations can be obtained by successive differentiations of Equation (1a) to give:

$$\begin{aligned} \left. \frac{dc}{dt} \right|_{t_n \leq t \leq t_{n+1}} &= X_n g^{(1)}(t-t_n) + \sum_{p=0}^{(q-1)} f_p^{(1)}(t-t_n) \left[\left. \frac{d^{(p)}c(t)}{dt^{(p)}} \right|_{t=t_n} \right] \\ \vdots & \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \end{aligned} \quad (1b)$$

$$\left. \frac{d^{(q-1)}c}{dt^{(q-1)}} \right|_{t_n \leq t \leq t_{n+1}} = X_n g^{(q-1)}(t-t_n) + \sum_{p=0}^{(q-1)} f_p^{(q-1)}(t-t_n) \left[\left. \frac{d^{(p)}c(t)}{dt^{(p)}} \right|_{t=t_n} \right] \quad (1c)$$

if we define

$$\left. \frac{d^{(i)} c(t)}{dt^{(i)}} \right|_{t=t_n} \triangleq C_n^{(i)}$$

and let $t=t_{n+1}$, the q equations reduce to the following set of difference equations:

$$C_{n+1} = (R_n - C_n)g(T_n) + \sum_{p=0}^{q-1} f_p(T_n)C_n^{(p)} \quad (2a)$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$C_{n+1}^{(q-1)} = (R_n - C_n)g^{(q-1)}(T_n) + \sum_{p=0}^{q-1} f_p^{(q-1)}(T_n)C_n^{(p)} \quad (2q)$$

where we have let $X_n = R_n - C_n$. Then the system is completely described by Equations (2a) to (2q).

The above equations can be replaced by an equivalent set which does not require knowledge of the derivatives of the output at the sampling instants but only the value of the output. The equations are written in the form:

$$C_{n+1} = (R_n - C_n)g(T_n) + \sum_{p=0}^{q-1} f_p(T_n)C_n^{(p)} \quad (3a)$$

$$C_n = (R_{n-1} - C_{n-1})g(T_{n-1}) + \sum_{p=0}^{q-1} f_p(T_{n-1})C_{n-1}^{(p)} \quad (3b)$$

$$\vdots \quad \quad \quad \vdots$$

$$C_{n-q+1} = (R_{n-q} - C_{n-q})g(T_{n-q}) + \sum_{p=0}^{q-1} f_p(T_{n-q})C_{n-q}^{(p)} \quad (3q)$$

By substitution, the derivatives at the sampling instants, $C_i^{(j)}$ can be eliminated, and an expression of the following form is obtained:

$$C_{n+1} = a_n(t)R_n + a_{n-1}(t)R_{n-1} + \dots + a_{n-q}(t)R_{n-q} + b_n(t)C_n + \dots + b_{n-q}(t)C_{n-q} \quad (4a)$$

where the a_i and b_i are functions of the past sampling interval durations T_i .

Example: Let $G(s)$ in Figure 1 be

$$G(s) = \frac{K}{s(s+1)}$$

The output during the n^{th} sampling interval then satisfies the equation

$$c(t) = KE_n \left[(t-t_n) - 1 + e^{-(t-t_n)} \right] + C_n + \dot{C}_n \left[1 - e^{-(t-t_n)} \right] \quad (t_n \leq t \leq t_{n+1}) \quad (4b)$$

which corresponds to Equation (1a). Differentiating once we obtain

$$\dot{c}(t) = KE_n \left[1 - e^{-(t-t_n)} \right] + \dot{C}_n e^{-(t-t_n)} \quad (4c)$$

Setting $t = t_{n+1}$ in Equations (8) and (9) one obtains

$$C_{n+1} = KE_n \left[T_n - 1 + e^{-T_n} \right] + C_n + \dot{C}_n \left[1 - e^{-T_n} \right] \quad (4d)$$

$$\dot{C}_{n+1} = KE_n \left[1 - e^{-T_n} \right] + \dot{C}_n e^{-T_n} \quad (4e)$$

From (4d) and (4e) one can also write, correspondingly,

$$C_n = KE_{n-1} \left[T_{n-1} - 1 + e^{-T_{n-1}} \right] + C_{n-1} + \dot{C}_{n-1} \left[1 - e^{-T_{n-1}} \right] \quad (4f)$$

$$\dot{C}_n = KE_{n-1} \left[1 - e^{-T_{n-1}} \right] + \dot{C}_{n-1} \left[e^{-T_{n-1}} \right] \quad (4g)$$

Terms containing \dot{C}_n and \dot{C}_{n-1} , can be eliminated using Equations (4d), (4f) and (4g). Then substituting $E_n = R_n - C_n$, we obtain:

$$\begin{aligned} C_{n+1} + \left[K(T_n - 1 + e^{-T_n}) - \frac{1 - e^{-(T_n + T_{n-1})}}{1 - e^{-T_{n-1}}} \right] C_n + \left[K(1 - e^{-T_{n-1}} - T_{n-1} e^{-T_{n-1}}) \right. \\ \left. + e^{-T_{n-1}} \right] \left(\frac{1 - e^{-T_n}}{1 - e^{-T_{n-1}}} \right) C_{n-1} = K(T_n - 1 + e^{-T_n}) R_n + K(1 - e^{-T_{n-1}} - T_{n-1} e^{-T_{n-1}}) \\ \left(\frac{1 - e^{-T_n}}{1 - e^{-T_{n-1}}} \right) R_{n-1} \end{aligned} \quad (5)$$

This is the desired equation, a difference equation with time varying coefficients, which yields the output at any sampling instant as a function of the values of the input and output at the two previous sampling instants and of the lengths of the two past sampling intervals. Two past samples are required since this is a second order system.

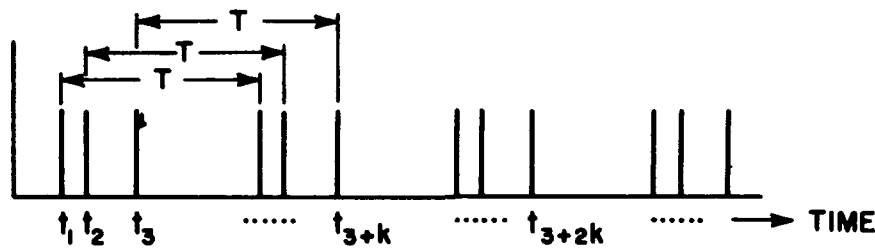
It should be noted that no restriction has been made, in the development of this equation, on the manner of obtaining the time intervals T_n . Jury and Mullin⁸ treat the case where the sampling pattern is periodic, repeating itself every k sample, i. e., $T_n = T_{n+k}$. In this case, one obtains difference equations with periodic coefficients which can be solved by means of z-transforms.¹⁴

Hufnagel⁶ considers the use of difference equations in the form here developed to perform a sample-by-sample transition between two regions where the sampling is periodic but at different frequencies. Standard z-transform techniques are used in the first region and the variable coefficient difference equations are used to obtain required initial conditions to make possible the use of z-transforms in the second region.

If the sampling period is signal-dependent, say on the magnitude of the error at the preceding sampling instant, the equations can still be used to obtain an exact transient solution to this nonlinear problem, proceeding sample-by-sample in the time domain. While this procedure is clearly very laborious, it can be easily mechanized for solution on a digital computer.

3. ANALYSIS USING MODIFIED z-TRANSFORMS

Modified z-transform techniques have been applied by several writers^{5, 19} to the analysis of sampled-data systems with a cyclic variation in sampling rate. It should be noted that in such systems the corresponding samples in each cycle are separated by the same time interval, as illustrated in Figure 2. Since a fundamental periodicity exists in cyclic-rate systems, it is not surprising that z-transform techniques are applicable. Two approaches to the problem will be illustrated below.

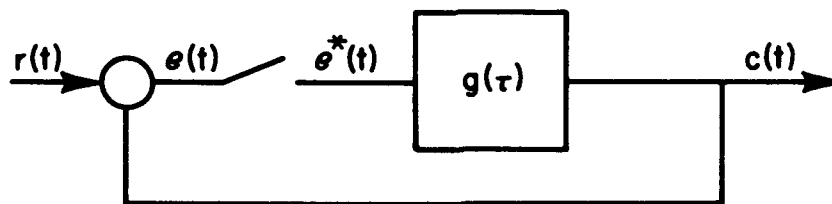


SAMPLING PATTERN IN A CYCLIC-RATE SYSTEM

FIGURE 2

Consider first the system of Figure 3 which does not include a hold circuit. Let the sampling times be t_1, t_2, \dots, t_n and the cycle time T be as shown in Figure 2, where there are n samples in each cycle pattern. Then, for a system with a basic sampling frequency $\frac{1}{T}$, the output can be obtained by recalling that the convolution summation for a linear sampled-data system can be expressed in the form

$$c(rT) = \sum_{k=0}^{\infty} g(kT) e(rT - kT) \quad (6)$$



ERROR-SAMPLED SYSTEM WITHOUT A HOLD CIRCUIT

FIGURE 3

Each of the samples in a cycle is delayed by t_i . Therefore, its response at time (rT) can be obtained by appropriately advancing the input function and delaying the weighting function $g(t)$, so that

$$c_i(rT) = \sum_{k=0}^{\infty} g(kT - t_i) e(rT - kT + t_i) \quad (7)$$

Since there are n such samples in a cycle, Equation (7) must be summed over the i 's. Finally, the output at time $(rT+t_j)$ is given by the appropriate advance in Equation (7), resulting in:

$$c(rT+t_j) = \sum_{i=1}^n \sum_{k=0}^{\infty} g(kT+t_j-t_i) e^{(rT-kT+t_i)} \quad (8)$$

By defining the "advanced z -transform" in the form

$$G_{ji}(z) = \sum_{k=0}^{\infty} g(kT+t_j-t_i) z^{-k} \quad (9)$$

the transform output becomes⁵

$$C_j(z) = \sum_{k=1}^n G_{ji}(z) E_i(z) \quad (10)$$

where

$$E_i(z) = \sum_{n=0}^{\infty} e(nT+t_i) z^{-n} \quad (11)$$

This latter form is related to the usual form $E(z, m)$ of the modified z -transform¹⁹ by the relation

$$E_i(z) = zE(z, m) \quad \text{if} \quad t_i = mT \quad (12)$$

Similarly, the quantities $G_{ji}(z)$ are related to the modified z -transforms by the relation

$$G_{ji}(z) = zG(z, m), \quad j \geq i \quad (13)$$

In order to express the system output in terms of the input $r(t)$ one can make use of the expression

$$E_k(z) = R_k(z) - C_k(z) \quad (14)$$

Substituting (14) into (10) one obtains:

$$\sum_{k=1}^n (\delta_{jk} + G_{jk}(z)) = \sum_{i=1}^n G_{ji}(z) R_i(z) \quad (15)$$

where δ_{jk} is the Kronecker delta function, i. e., $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ for $j \neq k$.

Hufnagel⁶ suggests that Equation (15) can be solved conveniently by considering the $G_{jk}(z)$ to be components of a matrix and $R_k(z)$ and $C_k(z)$ to be components of vectors, so that Equation (15) can be written

$$\begin{bmatrix} I + G(z) \end{bmatrix} \{C(z)\} = \begin{bmatrix} G(z) \end{bmatrix} \{R(z)\} \quad (16)$$

where I is the unit matrix. The output can be obtained by premultiplying by

$$\begin{bmatrix} I + G(z) \end{bmatrix}^{-1} :$$

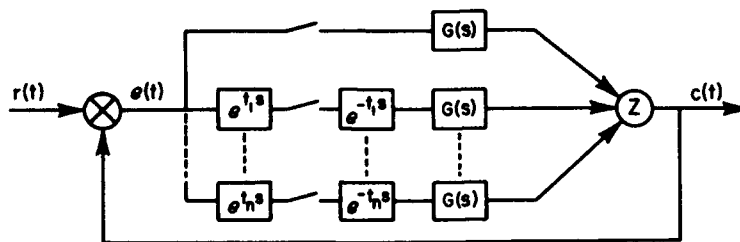
$$\{C(z)\} = \begin{bmatrix} I + G(z) \end{bmatrix}^{-1} \begin{bmatrix} G(z) \end{bmatrix} \{R(z)\} \quad (17)$$

The outputs at the sampling instants, $c(t)$, are obtained by evaluation of the inverse z -transform of the $C_k(z)$ by conventional methods.

It should be noted that when hold circuits are present, the matrix elements $G_{jk}(z)$ must be modified since the hold periods are of unequal length.

3.1 Modified z -Transforms from Switch Decomposition

The result obtained above for cyclic rate sampled-data systems can also be obtained in an easily visualized manner by following a procedure suggested by Tou.¹⁹ The cyclic-rate sampler of Figure 3 can be redrawn as shown in Figure 4 with n constant-rate samplers, for a system with n pulses per cycle.



EQUIVALENT DIAGRAM FOR CYCLIC-RATE SAMPLER

FIGURE 4

Each of the equivalent samplers in Figure 4 is represented as preceded by a fictitious advance element and followed by a fictitious delay element, the advances and delays being equal to the respective time interval between samples. If the elements preceding the sampler are denoted by $D_k(s)$,

$$D_k(s) = e^{t_k s}$$

and the elements following the sampler by $G_k(s)$,

$$G_k(s) = e^{-t_k s} G(s)$$

then the output can be obtained by summing the individual responses,

$$C(s) = \sum_{k=0}^{k=n} G_k(z) E D_k(z) \quad (18)$$

Taking the z-transform of Equation (18) we obtain

$$C(z) = \sum_{k=0}^{n-1} G_k(z) E D_k(z) \quad (19)$$

The terms $E D_k(z)$ are obtained from the relation

$$E(s) = R(s) - C(s) \quad (20)$$

by substituting $C(s)$ from Equation (18), multiplying by $D(s)$ and taking z-transforms. As a result one obtains a matrix equation

$$\begin{bmatrix} G D(z) \end{bmatrix} \{ E D(z) \} = \{ R D(z) \} \quad (21)$$

where the elements of the column vectors $\{ E D(z) \}$ and $\{ R D(z) \}$ are $E D_k(z)$ and $R D_k(z)$, respectively and the elements of the matrix

$\begin{bmatrix} G D(z) \end{bmatrix}$ are of the form:

$$\delta_{jk} + G_j D_k(z)$$

where

$$\delta_{jk} = 1 \quad \text{for } j = k \text{ and } \delta_{jk} = 0 \text{ for } j \neq k.$$

The solution follows lines analogous to those presented in the previous

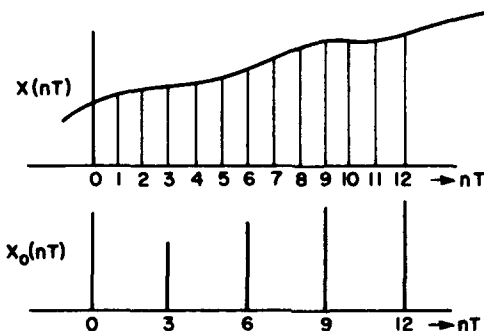
section and produces the modified z-transform of the output, $C(z, m)$.

It should be noted that the application of z-transforms is limited to cyclic variations in the sampling pattern, i. e., there must be a fundamental periodicity in the sampling. The technique has been extended to the multiple sampler case²¹ and to the systems with finite pulse width.⁹ If a multiple sampler system is subjected to a random input, it can be shown²⁰ that "Tuning" of one of the samplers to operate at an appropriate cyclic variable rate can reduce the mean-square error below that obtainable with constant rate sampling.

4. SKIPPED-SAMPLE PROBLEMS: TRANSFER MATRIX FORMULATION

In some systems samples may be missed occasionally, in an otherwise periodic constant sampling rate system. If the samples are missed at random, the problem may be treated by considering the sampling frequency itself to an appropriate random variable.¹ If the samples are missed at regular intervals, the problem reduces to that of a cyclic-rate system, which can be handled by the methods introduced in previous sections. In this section we shall consider an elegant solution to this problem, using a matrix approach to find the "skip-sampled components" of a signal, as presented by Friedland.⁴

Consider the sampling pattern shown in Figure 5, which also shows one skip-sampled component of the basic sampled signal. The signal is defined only at the sampling instants nT . The component $x_j(nT)$ of $x(t)$ is given by taking every Nth sample beginning with the jth (as shown in Figure 5 for $N = 3$, $j = 1$).



A SAMPLED SIGNAL AND THE ZERO-th SKIP-SAMPLED COMPONENT IF $N = 3$

FIGURE 5

Thus, the N skip sampled components are defined as

$$x_j(nT) = \begin{cases} x(nT), & n = (j + kN) \quad (j = 0, 1, \dots, N-1) \\ 0, & n \neq (j + kN) \quad k = 0, 1, \dots \end{cases} \quad (22)$$

Then the signal $x(nT)$ is represented as

$$x(nT) = \sum_{j=0}^{N-1} x_j(nT) \quad (23)$$

The z-transforms of these components are given by

$$X_j(z) = \sum_{n=0}^{\infty} x_j(nT) z^{-n} = \sum_{n=0}^{\infty} x\{(j+kN)T\} z^{-(j+kN)} \quad (24)$$

and

$$X(z) = \sum_{j=0}^{N-1} X_j(z) \quad (25)$$

Equation (25) may be considered as a column matrix representing every signal $x(t)$ in the system in terms of its skip-sampled components, i. e.,

$$\{X(z)\} = \begin{bmatrix} X_0(z) \\ X_1(z) \\ \vdots \\ X_{N-1}(z) \end{bmatrix} \quad (26)$$

The input-output relation for an arbitrary linear time-varying element in the time domain can be expressed as follows (at the sampling instants only):

$$\begin{bmatrix} y(0) \\ y(T) \\ y(2T) \\ \vdots \end{bmatrix} = \begin{bmatrix} h(0,0) & 0 & 0 & \dots \\ h(T,0) & h(T,T) & 0 & \dots \\ h(2T,0) & h(2T,T) & h(2T,2T) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} x(0) \\ x(T) \\ x(2T) \\ \vdots \end{bmatrix} \quad (27)$$

where $h(t, \tau)$ is the weighting function for the system, $\bar{x}(t)$ the input and $\bar{y}(t)$

the output vectors respectively, and $\bar{H} = [h(nT, kT)]$ is the transmission matrix of the system in the time domain. Now, since the skip-sampling is periodic, occurring every N samples, we have

$$h(nT+N, kT+N) = h(nT, kT) \quad (28)$$

The periodicity makes it possible to partition the transmission matrix into $(N \times N)$ submatrices and the column vectors into $(N \times 1)$ submatrices. Then, Equation (27) is written as follows:

$$\begin{bmatrix} \bar{y}(0) \\ \bar{y}(NT) \\ \bar{y}(2NT) \\ \vdots \end{bmatrix} = \begin{bmatrix} \bar{h}(0) & 0 & \dots \\ \bar{h}(NT) & \bar{h}(0) & \dots \\ \bar{h}(2NT) & \bar{h}(NT) & \dots \\ \vdots & \vdots & \dots \end{bmatrix} \begin{bmatrix} \bar{x}(0) \\ \bar{x}(NT) \\ \bar{x}(2NT) \\ \vdots \end{bmatrix} \quad (29)$$

In terms of these submatrices, the input-output relation can also be written as

$$\bar{y}(nNT) = \sum_{k=0}^n \bar{h}(nNT - kNT) \bar{x}(kNT) \quad (30)$$

which is the matrix equivalent of the ordinary convolution summation in the stationary case. By defining the z -transforms of the matrices of Equation (29), and modifying them with the appropriate powers of z , one obtains the matrix representation

$$\{Y(z)\} = [H(z)] \{X(z)\} \quad (31)$$

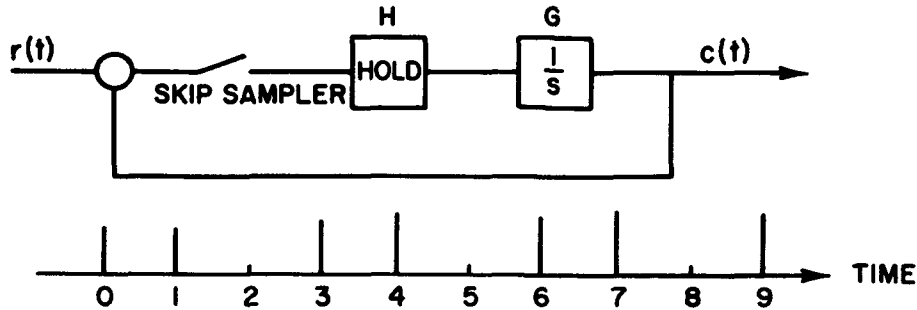
where $\bar{H}(z)$ is the transmission matrix in the frequency domain, or the "Transfer matrix". For a non-time-varying system, since $h(nT, kT) = h(nT-kT)$, the transfer matrix becomes:

$$\bar{H}(z) = \begin{bmatrix} H_0(z) & H_{N-1}(z) & \dots & H_1(z) \\ H_1(z) & H_0(z) & \dots & H_2(z) \\ \vdots & \vdots & \ddots & \vdots \\ H_{N-1}(z) & H_{N-2}(z) & \dots & H_0(z) \end{bmatrix} \quad (32)$$

where $H_1(z)$ is the pulse transfer function corresponding to the appropriate

skip-sampled transmission matrix. If the component is time-varying but has "zero-memory", then $H(z)$ reduces to a diagonal matrix, $\bar{A} = \text{diag} [a(0), a(T), a(2T), \dots, a[(N-1)T]]$. The construction of the matrices is best illustrated by an example from Friedland's paper.⁴

Consider the system of Figure 6, where the sampler closes for two sampling instants and skips one sampling instant. Thus, $N = 3$. The skip-



SKIP-SAMPLED SYSTEM AND SAMPLING PATTERN

FIGURE 6

sampling and hold can be represented by a separate matrix which takes into account the fact that, in general, the length of the hold period will depend on the sampling pattern. In this simple case, the hold is described by

$$G_{H_1}(s) = \frac{1 - e^{-Ts}}{s} \quad (33)$$

for sampling instants 0, 3, 6, ... and by

$$G_{H_2}(s) = \frac{1 - e^{-2Ts}}{s} \quad (34)$$

for sampling instants 1, 4, 7, ... If the integration is combined with the plant transfer function $\frac{1}{s}$, the hold can be represented by the matrix

$$\bar{H}(z) = \begin{bmatrix} 1 & -z^{-2} & 0 \\ -z^{-1} & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (35)$$

The plant, including the integration from the hold circuit, is

$$G^1(s) = \frac{1}{s^2}$$

so that

$$G(z) = \frac{z^{-1}}{(1-z^{-1})^2}$$

and

$$\bar{G}(z) = \begin{bmatrix} G_0(z) & G_2(z) & G_1(z) \\ G_1(z) & G_0(z) & G_2(z) \\ G_2(z) & G_1(z) & G_0(z) \end{bmatrix} \quad (36)$$

where the $G_i(z)$ are obtained by summing the respective skip-sampled components 0, 1 and 2 of $G(z)$ and are given by

$$G_0(z) = \frac{3z^{-3}}{(1-z^{-3})^2}; \quad G_1(z) = \frac{z^{-1} + 2z^{-4}}{(1-z^{-3})^2}; \quad G_2(z) = \frac{2z^{-2} + z^{-5}}{(1-z^{-3})^2} \quad (37)$$

The output of the system can be obtained by writing the overall transfer matrix for the system.

$$\bar{K}(z) = \bar{I} - (\bar{I} + \bar{G}\bar{H}(z))^{-1} \quad (38)$$

where I is the unit matrix. The response is then evaluated from

$$\bar{C}(z) = \bar{K}(z) \bar{R}(z) \quad (39)$$

4.1 Skipped-Sample Problems: Inversion by Long Division⁶

A less elegant but conceptually very simple method of obtaining the output of a skipped-sample system is based on the well known inversion of the output $C(z)$ by long division to obtain a series in z^{-n} . The coefficients of the series are the values of the output time function $c(nT)$ at the sampling instants.

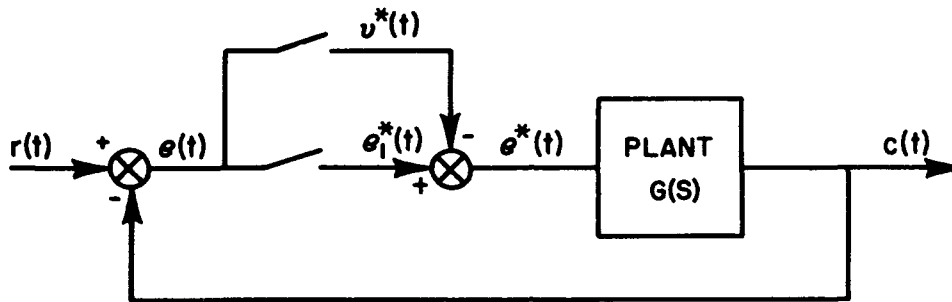
Consider the representation of Figure 7, where the skip-sampler is represented by a "negative sample generator", which cancels pulses at the "missed" sampling instants. Then, the transform of the system error can be written as

$$E(z) = \frac{R(z) + V(z)}{1 + G(z)} \quad (40)$$

where $V(z)$ is the transform of the "cancelling sampler" output, i. e.,

$$V(z) \triangleq \sum_{n=k} e(nT)z^{-n} \quad (41)$$

where k represents the values of the sampling instant n at which samples are omitted. If samples are omitted periodically, then $k = Nn$, where N represents the particular skipped sample; for example, $N = 3$ if every third sample is skipped.



REPRESENTATION OF SKIPPED-SAMPLE PROBLEM

FIGURE 7

Now Equation (40) could be used to obtain $e(nT)$ by long division, if $V(z)$ were known analytically. If it is not known, the process can still be carried out sample by sample. At each sampling instant where an error sample is omitted, a term corresponding to $V(z)$ is added to the numerator (or, more conveniently, to the remainder); at all other sampling instants $V(z)$ is zero and therefore can be omitted entirely.

If a zero-order hold circuit is present, the procedure must be modified since the output of the hold circuit is not zero following a "missing" sample, but rather will be the same as that due to the last existing sample pulse. This process can be represented by effectively sampling the last existing value again, i. e., if the n -th sample is omitted, we let

$$v(nT) = -\{e[(n-1)T] - e(nT)\} \quad (42)$$

Then

$$e(nT) = e(n-1)T$$

and the sampler output is equal to the pulse applied at $(n-1)T$ once again. The hold output will then remain constant through the "missed pulse" period.

Therefore, the function $v^*(t)$ can be represented as

$$v^*(t) = \sum_n \left\{ e(nT) - e[(n-k_n)T] \right\} \delta(t-nT) \quad (43)$$

where n represents the number of the missed samples and k_n the number of intervals since the last existing sample. The application of this method will be illustrated with a simple example.⁶

Let the "plant" $G(s)$ be of second order,

$$G(s) = \frac{2}{s(s+1)} \quad (44)$$

with no hold circuit used. Then

$$G(z) = \frac{1.264 z^{-1}}{(1-z^{-1})(1-0.368 z^{-1})} \quad (45)$$

Let the input be a unit step applied at $t = 0$. Then

$$R(z) = \frac{1}{1-z^{-1}} \quad (46)$$

and the error transform is

$$E(z) = \frac{R(z) - V(z)}{1+G(z)} = \frac{\left(\frac{1}{1-z^{-1}} \right) - V(z)}{1 + \frac{1.264 z^{-1}}{(1-z^{-1})(1-0.368 z^{-1})}} \quad (47)$$

Equation (47) reduces to

$$E(z) = \frac{(z^2 - 0.368 z) - V(z)(z-1)(z-0.368)}{(z^2 - 0.104 z + 0.368)} \quad (48)$$

Now, let the sampling pattern be such that every third sample is missed, as shown in Figure 5. Then the first term in $V(z)$ is $e(2T)z^{-2}$ and $V(z)$ can be omitted for the computation of the first two samples, as shown below

$$\begin{array}{r} z^2 - 0.104 z + 0.368 \left[\frac{1 - 0.264 z^{-1}}{z^2 - 0.368 z} \left[+ \frac{f(z) V(z)}{+ 0.368} \right] \right. \\ \hline - 0.264 z - 0.368 \\ - 0.264 z + 0.027 - 0.097 z^{-1} \\ \hline \left. - 0.395 + 0.097 z^{-1} \right] \end{array}$$

Now, since there is no sample at the instant $(2T)$, we add $v(2T)$ to the remainder and continue. Starting from the last remainder, we add $\left[-0.395 z^2(z-1)(z-0.368) \right]$:

$$\begin{array}{r}
 0 z^{-2} - 0.443 z^{-3} \\
 \hline
 z^2 - 0.104 z + 0.368 \left| \begin{array}{l} - 0.395 + 0.097 z^{-1} \\ - 0.395 + 0.540 z^{-1} - 0.146 z^{-2} \\ - 0.443 z^{-1} + 0.146 z^{-2} \\ - 0.443 z^{-1} + 0.046 z^{-2} - 0.163 z^{-3} \\ + 0.100 z^{-2} + 0.163 z^{-3} \\ \vdots \dots \dots \end{array} \right.
 \end{array}$$

Thus the first four terms of $e(nT)$ are then:

$$e(0) = 1.0, \quad e(T) = -0.264, \quad e(2T) = -0.395, \quad e(3T) = -0.443 \quad (49)$$

The output is then computed from the relation

$$c(nT) = r(nT) - e(nT) \quad (50)$$

It should be noted that while this method does not give the output in closed form directly, it is not limited to periodically skipped samples. The missed samples can be designated by an arbitrary function of (nT) in advance, or they can be computed as a function of system performance at the preceding intervals. Thus, for example, the sampler may be designed in such a manner that it samples at the instant $(n+1)T$ only if $e(nT) > \epsilon$, i. e., if the error exceeds certain limits. Similarly, the sampler operation may be governed by a performance criterion such as the sum of a number of terms of the mean-square error sequence. However, since the solution proceeds sample-by-sample from $t = 0$, it is probably limited to the investigation of the first few sampling intervals if it is not to become excessively time consuming.

5. THE STATE VARIABLE FORMULATION OF SAMPLING PROBLEMS

A formulation of sampled data problems using the concepts of state variables and state transition matrices has been presented by Kalman and Bertram.¹⁰ The method is primarily applicable to systems with periodic sampling (either constant-rate, cyclically varying rate, or multi-rate systems), and has been applied to the analysis of systems with random sampling.¹¹ As with other

methods, the generalization to arbitrary sampling periods makes the method extremely laborious. In fact, Kalman and Bertram emphasize that their approach is designed for use with digital computers.

The formulation of the method begins by dividing the components of linear sampled data systems into three groups: (a) continuous dynamic elements (characterized by linear differential equations); (b) discrete dynamic elements (characterized by linear difference equations); and (c) sample-and-hold elements which represent the transition from discrete to continuous information.

The state of the continuous elements is represented by the $(\gamma \times 1)$ matrix $\begin{bmatrix} \chi^c(t) \end{bmatrix}$ where γ is the sum of the orders of the differential equations governing these elements; the elements of $\begin{bmatrix} \chi^c(t) \end{bmatrix}$ are the values of the continuous quantities x_i and their derivatives at time t . The state of the discrete dynamic elements at time t is the set of numbers required at the start of a cycle of computation to solve the particular set of difference equations. They are denoted by a δ -vector $\begin{bmatrix} \chi^d(t) \end{bmatrix}$ if there are δ such numbers required.

The state of the sample and hold devices is denoted by a column matrix whose elements are the values of the outputs of these devices at time t . This is denoted by $\begin{bmatrix} \chi^s(t) \end{bmatrix}$ and is assumed to be of rank $(\sigma \times 1)$.

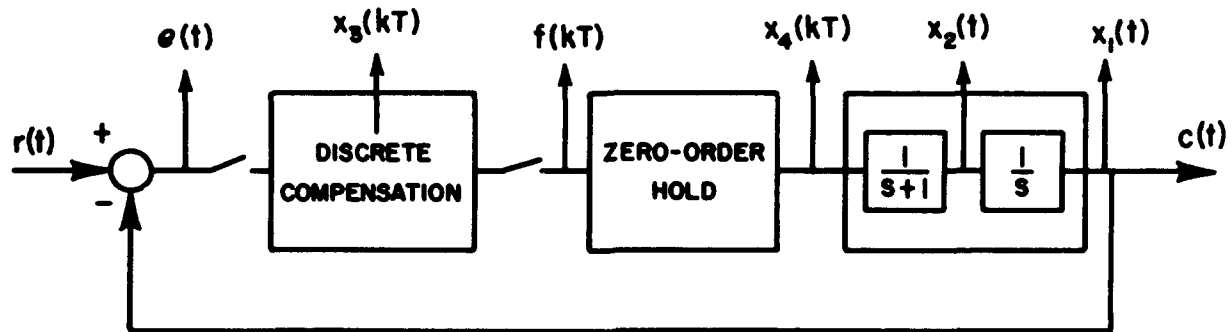
The state of the entire system is then obtained by combining the three state vectors into a single $(n \times 1)$ matrix which the authors call $\begin{bmatrix} \chi(t) \end{bmatrix}$, i. e.,

$$\{\chi(t)\} = \begin{bmatrix} \bar{\chi}^c(t) \\ \bar{\chi}^d(t) \\ \bar{\chi}^s(t) \end{bmatrix} \quad (51)$$

The transitions from sampling instant to sampling instant are then written by considering the state transitions of each type of element to be a matrix acting on the state variables.

The derivation of the transition matrices is beyond the scope of this report, but the method will be illustrated with one of Kalman and Bertram's examples. Consider the system of Figure 8, which includes discrete

compensation as well as a continuous "plant".



ILLUSTRATIVE PROBLEM FROM KALMAN AND BERTRAM¹⁰

FIGURE 8

The state variable vector for the continuous element is given by

$$\begin{bmatrix} \chi^c(t) \end{bmatrix} = \begin{bmatrix} \chi_1(t) \\ \chi_2(t) \end{bmatrix} = \begin{bmatrix} c(t) \\ \dot{c}(t) \end{bmatrix} \quad (52)$$

If the discrete compensator is described by the difference equation

$$f(t_k) = a_0 e(t_k) + a_1 e(t_{k-1}) - b_1 f(t_{k-1}) \quad (53)$$

then a new variable $\chi_3(t_k)$ can be defined by

$$\begin{aligned} f(t_k) &= a_0 e(t_k) + (a_1 - a_0 b_1) \chi_3(t_{k-1}) \\ \chi_3(t_k) &= e(t_k) - b_1 \chi_3(t_{k-1}) \end{aligned} \quad (54)$$

Thus, the state of the element at time t_k , given the input $e(t_k)$ is clearly described by

$$\begin{bmatrix} \chi^D(t_k) \end{bmatrix} = \begin{bmatrix} \chi_3(t_{k-1}) \end{bmatrix} \quad (55)$$

The state of the sample-and-hold element is simply its output, which, at the sampling instants, equals its input:

$$\begin{bmatrix} x^s(t_k) \end{bmatrix} = \begin{bmatrix} x_4(t_k) \end{bmatrix} - \begin{bmatrix} f(t_k) \end{bmatrix} \quad (56)$$

In the above equations, t_k is the k th sampling instant after $t = 0$, and $T_k = (t_{k+1} - t_k)$.

Now it can be shown that the state of the complete system at time t_{k+1} is described by the equation

$$\{x(t_{k+1})\} = \begin{bmatrix} \psi(T_k) \end{bmatrix} \{x(t_k)\} + \begin{bmatrix} \nu(T_k) \end{bmatrix} r(t_k) \quad (57)$$

where

$\{x(t_{k+1})\}$ is the state vector at time t_{k+1}

$\begin{bmatrix} \psi(T_k) \end{bmatrix}$ is the system transition equation from instant t_k to instant t_{k+1}

$r(t_k)$ is the input at time t_k (a scalar)

$\begin{bmatrix} \nu(T_k) \end{bmatrix}$ is a vector including the effect of the input on the state transition.

For the particular system of this example, the transition matrix is given by:

$$\begin{bmatrix} \psi(T_k) \end{bmatrix} = \left[\begin{array}{cc|cc} 1 - a_0(T_k^{-1} + e^{-T_k}) & 1 - e^{-T_k} & (T_k^{-1} + e^{-T_k})(a_1 - a_0 b_1) & 0 \\ -a_0(1 - e^{-T_k}) & e^{-T_k} & (1 - e^{-T_k})(a_1 - a_0 b_1) & 0 \\ \hline -1 & 0 & -b_1 & 0 \\ \hline -a_0 & 0 & a_1 - a_0 b_1 & 0 \end{array} \right] \quad (58)$$

and

$$\begin{bmatrix} \nu(T_k) \end{bmatrix} = \left[\begin{array}{c} a_0(T_k^{-1} + e^{-T_k}) \\ a_0(1 - e^{-T_k}) \\ \hline 1 \\ \hline a_0 \end{array} \right] \quad (59)$$

The top two rows of the matrices in Equations (58) and (59) govern the state transitions of the continuous elements; the third row concerns the discrete compensator and the fourth row the zero-order hold.

While the formulation of the equations in matrix form is very elegant, it should be noted that Equation (57) is merely a concise statement of the difference equation method outlined in a previous section of this report. The matrix formulation has particular appeal in the stationary case. Even in the periodically varying sampling rate case, when the problem is stated in terms of difference equations with periodic coefficients, the "transition matrix" becomes stationary if attention is focused on the transition corresponding to a complete period of sampling operations.

6. OTHER METHODS

6.1 Continuous Approximations

If the sampling frequency ω_s is sufficiently high compared to the band-pass of the plant $G(j\omega)$, then the sample-and-hold operation can be approximated with one or two terms of its frequency domain representation. A zero-order hold is described by

$$H(j\omega) = T e^{-j\omega \frac{T}{2}} \frac{\sin \omega T/2}{\omega T/2} \quad (60)$$

Therefore, the pulse transfer function of a sampler followed by a zero-order-hold and plant $G(j\omega)$ is given by

$$HG^*(j\omega) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} T e^{-jT(\omega + n\omega_s)/2} \frac{\sin[(\omega + n\omega_s)T/2]}{(\omega + n\omega_s)T/2} G(j\omega + jn\omega_s) \quad (61)$$

Since for low frequencies $\frac{\sin \omega T/2}{\omega T/2} \approx 1$ and the higher harmonics are assumed negligible, this transfer function reduces to

$$HG^*(j\omega) \approx e^{-j\omega T/2} G(j\omega) \quad (62)$$

This approximation is the foundation of the most common approximation methods.² Linvill^{12, 13} has suggested the extension of the approximation by using

one or two of the additional sidebands due to sampling. Clearly, the difficulty with applying this type of approximation to the variable sampling rate problem is that one obtains, at the very least, continuous control systems with variable time delays. The solution of problems of this type is also at best approximate. Other continuous design methods^{7, 16} suffer from similar limitations. Rubtsov¹⁷ has presented a careful analysis of the validity of some of the approximation methods.

6.2 Slowly-Varying Sampling Rate Systems

If the sampling rate varies slowly in comparison to the dominant system time constants, other approaches are possible. (The concept of "slow variation" can be related to the change in the sampling rate during the time required for the impulse response of the system to decay below a specified value.) Tartakovskii¹⁸ has applied the methods used by Zadeh for the study of stability of time varying linear equations to the stability of difference equations with varying coefficients. This method makes it possible to apply z-transform techniques to obtain approximate stability conditions for slowly varying systems.

If the sampling rate varies slowly it may also be considered piecewise constant. Ordinary z-transforms can be used for the solution during a constant-rate period, and the difference equations of the system solved step-by-step at transitions of sampling frequency to establish initial conditions.⁶

7. SUMMARY AND CONCLUSION

This report has presented a survey of methods presently available for the analysis of sampled data systems with a variable sampling rate. A number of exact methods can be used for systems where the sampling rate varies periodically, since the inherent periodicity makes the use of z-transforms possible.

The general time-dependent sampling interval problem can be solved by writing the state transition equations of the system from interval to interval. These equations can be expressed in concise form using matrices, but

except in extremely simple cases, they are too complex and time consuming for solution except with a digital computer. A number of the methods presented are practical for evaluation of system response during the first few sampling instants, even if the sampling intervals are given by arbitrary functions of time.

When the sampling rate is dependent on system performance the problem becomes nonlinear. Recursion equations for solution of the nonlinear difference equations can still be written and the above remarks apply. Most of the other methods surveyed in this report cannot be extended directly to the nonlinear case.

While stability evaluation was not discussed in the report, Kalman and Bertram¹⁰ indicate the extension of stability criteria from constant-rate to periodically varying sampling rate systems. No general results appear to be available for the arbitrary variable-rate sampled-data system.

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Unclassified Report

This report presents several techniques which can be used for the analysis of sampled-data systems with a non-constant sampling period. It is shown that the application of z-transform techniques is limited to cases where the sampling pattern repeats periodically. Several special cases,

(over)

including the cyclically-varying sampling period and the "Skipped sample" problem are outlined. The most general methods available are based on a direct solution of the system difference equations. These equations, while time consuming to solve, do make possible the evaluation of transient response sample-by-sample. The advantages, limitations and possible extensions of the various methods are outlined. The report includes a number of simple examples and an extensive bibliography.

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